Dynamics of solitary waves in the Zakharov model equations

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We analyze internal vibrations of a solitary wave in the generalized Zakharov system (including a direct nonlinear self-interaction of the high-frequency field) by means of a variational approach. The application of the variational approximation to this model turns out to be nontrivial, as one needs to renormalize the Lagrangian in order to avoid divergences. This is done with the use of two fundamental integrals of motion of the model. We derive a Hamiltonian two-degrees-of-freedom dynamical system that governs internal vibrations of the solitary wave. The eigenfrequencies of the small oscillations around the unperturbed solitary wave are found explicitly, one of them lying inside the gap of the high-frequency subsystem, the other one being well above the gap. Finite-amplitude oscillations are simulated numerically. It is shown that these oscillations remain regular if the perturbation does not break the balance between the two integrals of motion, while in the opposite case the oscillations are more irregular and may possibly become chaotic. [S1063-651X(97)01601-2]

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I. INTRODUCTION

The Zakharov system of equations [1] (ZS) is one of the fundamental models governing dynamics of nonlinear waves in one-dimensional systems. It describes, in a general form, the interaction between high-frequency (hf) and low-frequency (lf) waves. The physically most important example involves the interaction between the Langmuir and ion-acoustic waves in plasmas [1]. Other physical applications are also possible, for example, the interaction of hf and lf perturbations in the atmosphere [2]. In contrast with some other fundamental one-dimensional nonlinear wave models, the ZS is not integrable [3], although it has a stable single-solitary-wave solution, which in the rest of this paper will be referred to as a *soliton*. This solution plays a most important role in many applications. As usual, the existence of the soliton is related to the modulational instability of the continuous wave in this model [4].

Since the system is nonintegrable, the dynamics of the fundamental soliton is nontrivial even in the absence of additional perturbations. In early works [5], the interaction of the soliton with acoustic (lf) wave packets was simulated numerically. It was shown that the interaction may be essentially inelastic. A typical manifestation of the inelasticity is that an acoustic wave packet, colliding with the soliton, decreases its energy by expelling a part of the quanta (plasmons) that are bound inside the soliton. Later, a perturbation theory for the description of this interaction was developed [6]. A perturbation analysis was presented for the nearly adiabatic case, when the ZS could be approximated by a perturbed nonlinear Schrödinger (NLS) equation.

Recently, it was demonstrated that the interaction of the soliton with a random acoustic field in the ZS gives rise to another effect, which is simpler and maybe even more important for applications than the inelastic interaction (which in this case is the emission of radiation from the soliton induced by the random acoustic field), viz., a random walk of the soliton [7]). Collisions between solitons are also inelastic in the ZS. The collisions were simulated in detail in Ref. [8] within the framework of the generalized ZS, which included a direct nonlinear self-interaction in the hf subsystem and also two different acoustic fields with different sound velocities. It was demonstrated that the collisions not only are accompanied by emission of radiation, but also may result in a fusion of the two solitons into a ''breather,'' i.e., a single soliton with strong internal vibrations.

Another fundamental manifestation of the nonintegrability of the ZS is the possibility of the existence of a dynamical chaos. Recently, spatiotemporal chaos was discovered in direct simulations of the ZS with periodic boundary conditons [9]. A transition to chaotic states from regular quasiperiodic regimes was also considered [9].

There is still another fundamental dynamical process that so far has not been considered in the ZS, which will be addressed in the present work, namely, internal vibrations of the soliton. Internal vibrations are observed in numerical simulations of solitons in many integrable (as well as in nonintegrable) models. The vibrations are easily excited, both in nonintegrable and integrable systems, by a deviation of the initial pulse from the exact soliton and, in nonintegrable models, as a result of collisions between the solitons. As it was demonstrated in Ref. [8], collisions in the generalized ZS always give rise to a strong excitation of the soliton's internal oscillations, even if the collision does not lead to fusion into a single soliton.

In general, the vibrations are gradually damped by radiative losses. However, in many cases the rate of emission of radiation turns out to be very small, so that the vibrations may be fairly persistent. They were first identified as a dis-

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tinct dynamical phenomenon in numerical simulations of the usual NLS equation [10]. Then, an effective analytical approximation for the description of the soliton internal vibrations was elaborated [11]. This approximation is based on the variational principle. One assumes that the soliton keeps a fixed functional form (the so-called ansatz), but that its parameters are allowed to vary in time. Inserting the ansatz into the Lagrangian of the underlying model, one can explicitly integrate it over the spatial variable. Thus one arrives at an *effective* Lagrangian for the free parameters of the ansatz. This, in turn, leads to a finite-dimensional dynamical system, which furnishes the simplified description of the dynamics of the system.

Of course, the accuracy of this approximation must be monitored by comparison with direct simulations, as the technique is approximate (first of all, the emission of radiation is completely ignored). Nevertheless, this simple approximation yields surprisingly accurate results even in really complicated problems, such as, e.g., soliton propagation in a nonlinear optical fiber with a periodically modulated dispersion [12,13]. The variational approximation has predicted that the soliton will be destroyed suddenly if the modulation depth exceeds a certain critical value (which is actually rather small) [12]. The destruction was expected in the form of an unlimited growth of the width of the soliton, i.e., its direct decay into dispersive radiation (it should be noted that, in this form, the variational approach can sometimes indirectly describe an essentially radiative process). In the case when the modulation depth is smaller than the critical value, the soliton was predicted to exist stably in the form of a *breather*, i.e., with persistent internal oscillations. Later, detailed simulations have confirmed that the actual behavior of the soliton is quite close to this prediction [13]. An essential difference was, however, that the actual destruction mode was not the decay into radiation, but splitting of the soliton into two secondary ones, which was indeed accompanied by a burst of radiation. The simplest variational approximation employed in Ref. [8] could not predict this particular destruction mode simply because the underlying ansatz was based on a single soliton. Nevertheless, the actual value of the critical depth of the modulation at which the destruction takes place differs by not more than 15% from the predicted value. Many other features predicted by the variational approximation were also confirmed by the simulations, for instance, the important fact that the critical modulation depth very weakly depends upon the soliton energy and that the surviving soliton is a vibrating breather.

The objective of the present work is to analyze internal oscillations of the soliton in the Zakharov system by means of the variational approximation. Besides its obvious importance for applications, this problem also turns out to be quite interesting in itself, as one must essentially modify some elements of the variational technique in order to be able to apply it to ZS. The main technical difficulty that we encounter when applying the technique to the soliton in the ZS is the divergence of a ‘‘naively defined’’ Lagrangian in this model. We demonstrate that one can resolve this problem by directly using two fundamental integrals of motion of the ZS.

The variational approach will be described, with necessary technical details, in Sec. II. In Sec. III, we present analytical results for small internal vibrations of the soliton. In the linear approximation we find two different eigenfrequencies of these vibrations. The eigenfrequencies remain always real, which implies that the soliton is always stable. This is in full agreement with the results of numerical simulations of the ZS.

It is relevant to notice here that eigenmodes of the small oscillation predicted by the variational approximation, may, generally speaking, correspond either to the ‘‘quasimodes’’ in the exact equations, which exist persistently but gradually decay into radiation because they directly couple to the continuous spectrum of the system (the quasimode of the NLS soliton is a well-known example [11]), or to a genuine discrete eigenmode in the linear spectrum of the small vibrations around the soliton. The latter eigenmode must belong to a gap of the continuous spectrum, provided that such a gap exists. In the case of the ZS, genuine eigenmodes are not possible because the lf continuum has no gap.

In Sec. IV, we display the results of numerical simulations of finite-amplitude oscillations governed by the variational equations. The amplitude of the oscillations is always limited: above a certain maximum amplitude, the soliton width, instead of oscillating, will start to grow, which implies decay into radiation [12]. Below the critical value, the character of the oscillations depends strongly upon the values of the integrals of motion of the excited soliton. The ZS has, besides the Hamiltonian and momentum, two additional integrals of motion: the hf ‘‘number of quanta’’ and the lf ‘‘mass.’’ In a balanced state, in which these two integrals of motion are related the same way as for the unperturbed soliton (i.e., they are simply equal), the simulations demonstrate that the oscillations are always periodic or quasiperiodic. In an unbalanced state, when this relation between the two integrals is broken, the motion becomes more complicated and possibly also chaotic; cf. [9].

II. THE VARIATIONAL APPROXIMATION

In this work, we consider the generalized Zakharov equations for the complex envelope $u(x,t)$ of the hf wave and for the real lf field $n(x,t)$. Thus

$$iu_t+u_{xx}-2\lambda|u|^2u+2nu=0, \tag{1}$$

$$n_{tt}-n_{xx}=-(|u|^2)_{xx}, \tag{2}$$

where the sound velocity, as well as the coupling constant in Eq. (2), has been normalized to unity. The cubic term in Eq. (1) describes direct nonlinear self-interaction in the hf subsystem. In plasma physics, such a term can arise with $\lambda<0$ due to relativistic effects (the velocity dependence of the electron mass [14]) and it then corresponds to a self-focusing effect. In the analysis following below we will consider a coefficient λ with either sign.

It is easy to see that Eqs. (1) and (2) have a family of exact soliton solutions of the form

$$u(x,t)=(1-\lambda)^{-1/2}\eta\,\mathrm{sech}(\eta x)e^{i\eta^2 t}, \tag{3}$$

$$n(x,t)=(1-\lambda)^{-1}\eta^2\mathrm{sech}^2(\eta x), \tag{4}$$

where η is an arbitrary amplitude of the soliton (in this work, we consider only quiescent solitons, although an exact solution for moving ones is well known too). To introduce the Lagrangian of the ZS, one has to define the so-called potential $\nu(x,t)$ from

$$n\equiv\nu_{xx}. \tag{5}$$

In terms of this potential, the Lagrangian density corresponding to Eqs. (1) and (2) takes the form

$$\mathcal{L}=\frac{1}{2}i(u^*u_t-uu_t^*)-|u_x|^2-\lambda|u|^4+2\nu_{xx}|u|^2+\nu_{xt}^2-\nu_{xx}^2. \tag{6}$$

The full Lagrangian L is defined as the integral of the Lagrangian density

$$L=\int_{-\infty}^{+\infty}\mathcal{L}dx. \tag{7}$$

It is straightforward to see that Eqs. (1) and (2) have four integrals of motion: the momentum (which will not be used in this work), the Hamiltonian

$$H=\int_{-\infty}^{+\infty}(|u_x|^2+\lambda|u|^4-2\nu_{xx}|u|^2-\nu_{xt}^2+\nu_{xx}^2)dx$$

(which, as a matter of fact, will not be explictly used either), the number of hf quanta

$$W=(1/\sqrt{\pi})\int_{-\infty}^{+\infty}|u|^2dx, \tag{8}$$

and the lf mass

$$C=(1/\sqrt{\pi})\int_{-\infty}^{+\infty}ndx. \tag{9}$$

The multiplier $1/\sqrt{\pi}$ in these definitions will render the use of the integrals (8) and (9) more convenient in what follows below.

A crucial role in the success or failure of the variational approximation is played by the choice of the ansatz. In this work, we are dealing with the two-component model (1) and (2), in which we must allow the two components of the soliton to have different widths. As it is well known, the only analytically tractable ansatz that admits this property is based on a Gaussian shape. Accordingly, we adopt the following ansatz for the two-component soliton:

$$u(x,t)=A\exp\left(-\frac{x^2}{2a^2}+i\phi+ikx^2\right), \tag{10}$$

$$n(x,t)=B\exp\left(-\frac{x^2}{b^2}\right). \tag{11}$$

Here A and B are the amplitudes of the two components, a and b are their widths (notice the difference in defining the widths for the two components), ϕ is the phase of the hf envelope, and k is the so-called chirp, which is necessary to provide a balance between the different dynamical variables [11]. All the six free parameters in the ansatz (10) and (11) are real and they are all assumed to be functions of time only.

The next step, before proceeding to the effective Lagrangian, is to calculate the integrals of motion (8) and (9) for the ansatz (10) and (11). This yields

$$W=A^2a,\quad C=Bb. \tag{12}$$

In the usual application of the variational approach to the description of internal vibrations of the soliton [11], first the ansatz is inserted into the Lagrangian density and then the conservation laws are obtained among other equations generated by variation of the resultant effective Lagrangian. In all the cases, the conserved quantities are obtained exactly in the same form that is produced directly by insertion of the ansatz into the exact integral expressions, cf. Eqs. (12). Moreover, it is also known that the eventual variational equations take the same form if, from the very beginning, one treats the combinations of the ansatz parameters that constitute the conserved quantities, e.g., (12), as arbitrary constants, thus eliminating some of the variational parameters. This trick significantly simplifies the derivation of the final dynamical equations and will play a crucial role in the present case.

Because the Lagrangian density (6) is written in terms of the potential ν instead of the physical field n, one needs to know the function $\nu(x)$ corresponding to the ansatz (11). It is straightforward to see that direct calculation of the effective Lagrangian would give rise to a divergence produced by the term ν_{xt}^2 in the Lagrangian density (6). A way to circumvent the divergence is to use the conserved quantities in order to eliminate the amplitudes A and B, i.e., $A^2\equiv W/a$ and $B\equiv C/b$.

For the derivative ν_{xt} that enters the Lagrangian, we obtain the following expression from Eqs. (5) and (11):

$$\nu_{xt}=Cb^{-2}\dot{b}\int_0^x(2y^2/b^2-1)e^{-y^2/b^2}dy \tag{13}$$

where the overdot denotes the time derivative. It follows from Eq. (13) that $\nu_{xt}\to 0$ at $x\to\pm\infty$, which will now provide for convergence of the effective Lagrangian.

Finally, the effective Lagrangian takes the form

$$\pi^{-1/2}L=-W\dot{\phi}-Wa^2\dot{k}-\frac{1}{2}W(4k^2a^2+a^{-2})-\frac{1}{\sqrt{2}}\lambda W^2a^{-1}$$
$$+2CW(a^2+b^2)^{-1/2}+IC^2b\dot{b}^2-\frac{1}{\sqrt{2}}C^2b^{-1}, \tag{14}$$

where I is a numerical coefficient given by

$$I=\frac{2}{\sqrt{\pi}}\int_{-\infty}^{+\infty}dx\int_0^x dy\int_0^x dz(2y^2-1)(2z^2-1)e^{-(y^2+z^2)}$$
$$\approx 0.1567.$$

A full system of equations for the variational parameters can immediately be obtained by varying this Lagrangian. First,

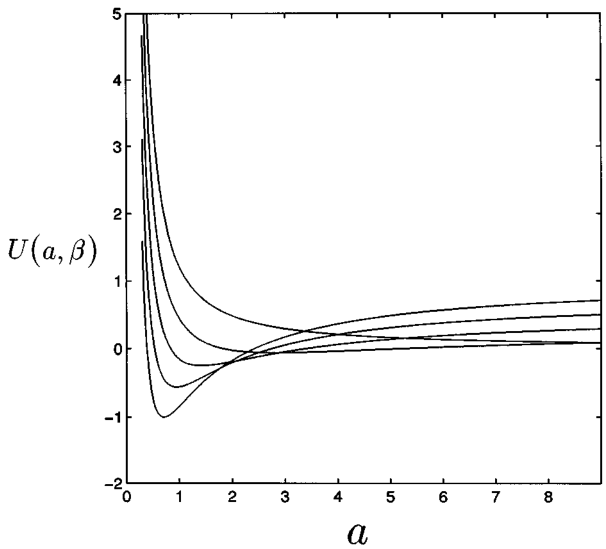

FIG. 1. Set of cross sections of the effective potential $U(a,\beta)$, defined by Eq. (19), for different values of λ ($\lambda=-1,-0.5,0,0.5,0.999$). The sections are taken at $\beta=\beta_0$, where β_0 corresponds to the fixed point (20); $W=C=1$.

the variation in ϕ reproduces the result that W is a constant. Next, the variation in k yields the usual expression for the chirp

$$k=\dot{a}/2a. \tag{15}$$

Eliminating k by means of this relation, the equations describing the evolution of a and b can be finally written in the form

$$\ddot{a}=a^{-3}+\frac{1}{\sqrt{2}}\lambda Wa^{-2}-2Ca(a^2+\beta^{4/3})^{-3/2}. \tag{16}$$

$$\frac{8IC}{9W}\ddot{\beta}=-\frac{4}{3}\beta^{1/3}(a^2+\beta^{4/3})^{-3/2}+\frac{\sqrt{2}C}{3W}\beta^{-5/3}, \tag{17}$$

where $\beta\equiv b^{3/2}$. These equations are analogous to those for particle motion with two degrees of freedom a and β and an anisotropic mass with the components

$$m_a=1, \quad m_\beta=8IC^2/9W \tag{18}$$

in the external potential

$$U(a,\beta)=\frac{1}{2}a^{-2}+\frac{1}{\sqrt{2}}\lambda Wa^{-1}-2C(a^2+\beta^{4/3})^{-1/2}$$
$$+\frac{1}{\sqrt{2}}C^2W^{-1}\beta^{-2/3}. \tag{19}$$

A full two-dimensional picture showing the shape of the potential (19) turns out to be not very useful, and instead Fig. 1 shows only a set of cross sections of the potential for different values of λ. These cross sections are taken for constant β and are going through the fixed point of Eqs. (16) and (17). In the next section, we will consider small oscillations around the fixed point (FP) of the dynamical system (16) and (17).

III. SMALL OSCILLATIONS

A FP of the dynamical system governing the vibrations of the soliton corresponds to an unperturbed soliton. A straightforward calculation shows that the exact solution (3) and (4) corresponds to equal values of the integrals of motion (8) and (9). Therefore, we will be looking for a FP of the dynamical system (16) and (17) imposing the same constraint $C=W$. An elementary consideration shows that in this case there exists a single physical FP at

$$a_0^{-1}=\beta_0^{-2/3}=\frac{1-\lambda}{\sqrt{2}}W. \tag{20}$$

Notice that, in terms of the original parameter b, this FP has $a_0=b_0$, i.e., in accordance with the exact solution (3) and (4), it predicts equal widths of both components of the unperturbed soliton. It is also noteworthy that both the exact solution and the FP (20) have the same existence condition $\lambda<1$.

The next natural step is to consider small oscillations in a vicinity of the FP. Linearization of Eqs. (16) and (17) around the FP leads to the following equation for the eigenfrequency ω of the small oscillations:

$$\left[a_0^4\omega^2-\frac{5-2\lambda}{2(1-\lambda)}\right]\left[a_0^4\omega^2-\frac{5}{4\sqrt{2I}}\right]-\frac{9}{8\sqrt{2}I(1-\lambda)}=0. \tag{21}$$

One can readily check that both roots $\omega_{1,2}^2$ of the biquadratic equation (21) satisfy the stability condition $\omega^2>0$ at all values of λ at which the FP exists.

It is interesting to compare the eigenfrequencies of the small oscillations with the frequency $\omega_0\equiv-\dot{\phi}$ of the unperturbed soliton, see Eq. (10). Within the framework of the variational approach, this frequency can be obtained by varying the Lagrangian (14) with respect to the parameter W with all the other parameters taken at the FP. This yields

$$\omega_0=-\frac{1}{2}a_0^{-2}+\sqrt{2}(1-\lambda)Wa_0^{-1}\equiv-\frac{3}{4}(1-\lambda)^2W^2. \tag{22}$$

The quantities $\omega_{1,2}^2$ given by Eq. (21) and ω_0^2 given by Eq. (22) are shown, as functions of the parameter λ, in Fig. 2.

An interesting issue is the radiative damping of the internal vibrations of the perturbed soliton. In the ZS, the soliton can emit radiation through both the hf and lf channels. Although we will not consider in this work the emission problem in detail, some comments are in order. In the lf channel, the emitted waves have the acoustic dispersion law with no gap. Therefore, shape vibrations of the soliton with any value of the frequency directly give rise to emission of the lf (acoustic) waves. Contrary to this, it is well known [15] that a perturbation of the soliton with a frequency ω directly induces emission of the hf waves only if $\omega^2>\omega_0^2$, i.e., if the perturbation frequency is above the effective gap induced by

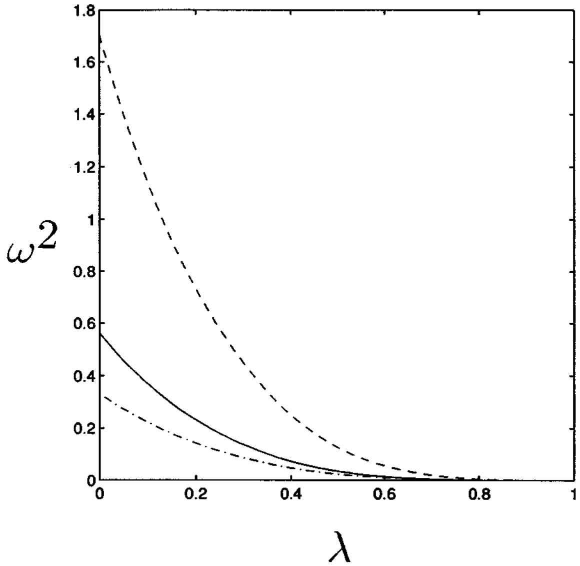

FIG. 2. Squared eigenfrequencies of small-amplitude oscillations around the fixed point (20), according to Eq. (21) (the dash-dotted and dashed lines), and the squared frequency (22) of the unperturbed soliton (the solid line).

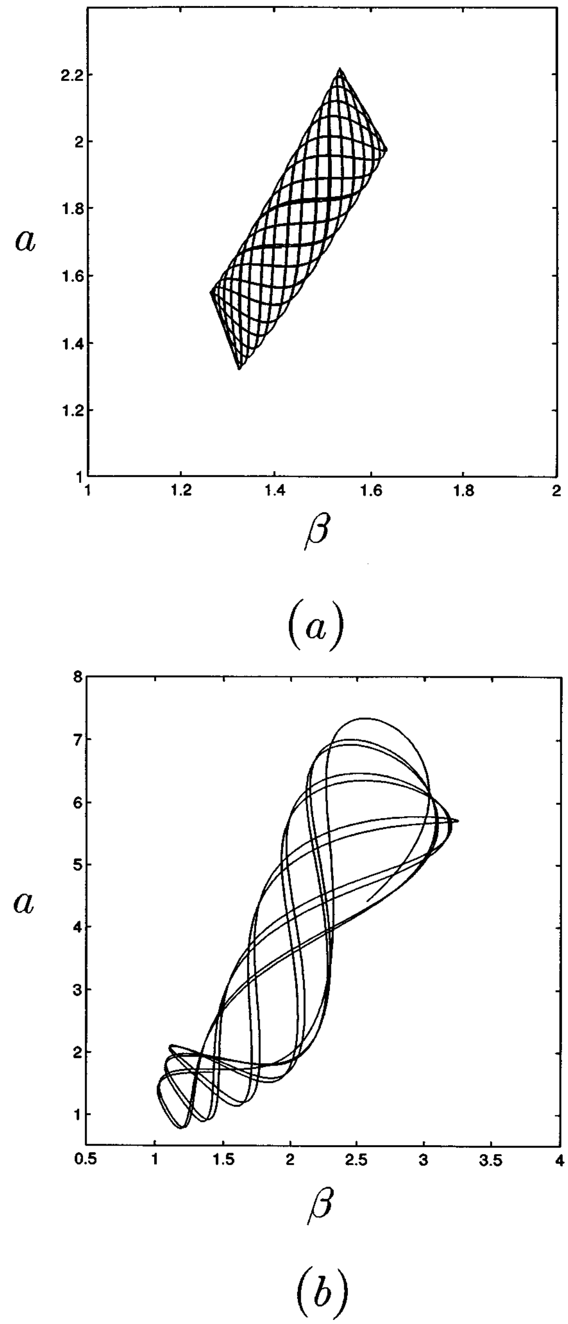

FIG. 3. Set of typical dynamical trajectories of the system (16) and (17) with $\lambda=0$ and $W=C=1$, for different values of the initial velocity $v_0 \equiv \dot{a}(t=0)$: (a) $v_0=-0.1$ and (b) $v_0=-0.4$.

the internal frequency ω_0 of the unperturbed soliton. Looking at Fig. 2, one concludes that one of the eigenfrequencies of the perturbed soliton lies inside the hf gap, while another is far above it.

IV. LARGE-AMPLITUDE OSCILLATIONS

As it was discussed in the Introduction, in many cases the variational approximation furnishes quite an accurate description of strong perturbations of a soliton. In this section, we will consider finite-amplitude shape vibrations of the soliton in ZS. The consideration will be based on numerical simulations of the dynamical system (16) and (17). A comparison with direct simulations of the partial differential equations (1) and (2) is deferred to another work.

In Fig. 3, we display a series of plots presenting typical dynamical trajectories of the system (16) and (17) with $\lambda=0$ and with the balanced integrals of motion, i.e., $W=C$. In all the cases, the initial point is taken at the FP, but the system is given a certain initial ''velocity'' in the direction of a, viz., $v_0 \equiv \dot{a}(t=0)$. Actually, this velocity represents an initial chirp of the soliton according to Eq. (15). As it is evident from Fig. 3, for all values of the initial velocity v_0 such that $|v_0|<v_{\mathrm{crit}}$, where v_{crit} is the critical initial velocity at which the particle is kicked out of the potential well (i.e., the soliton is expected to decay into radiation [12]), the oscillations remain quite regular: periodic or quasiperiodic. It is clearly seen that the frequency of the oscillations strongly decreases with increasing amplitude. This anharmonism is a direct consequence of the shape of the effective potential (19) (see Fig. 1).

In Fig. 4, a similar series of plots is shown for $\lambda=-0.3$ and 0.3. In these cases, there is an additional initial perturbation besides the velocity, namely, the initial point is taken at the FP corresponding not to the actual value of λ but to $\lambda=0$. The oscillations remain regular in this case too.

We note that the condition determining the critical velocities in the a and β directions, at which the particle escapes the potential well, can be obtained from the conservation of the Hamiltonian of Eqs. (16) and (17),

$$H=\frac{1}{2}m_a\dot{a}^2+\frac{1}{2}m_\beta\dot{\beta}^2+U(a,\beta), \tag{23}$$

where the effective masses are given by Eq. (18). Evaluating the Hamiltonian for the case when the particle barely escapes ($\dot{a}\to 0, \dot{\beta}\to 0$ as $a\to\infty, \beta\to\infty$), we obtain the relation

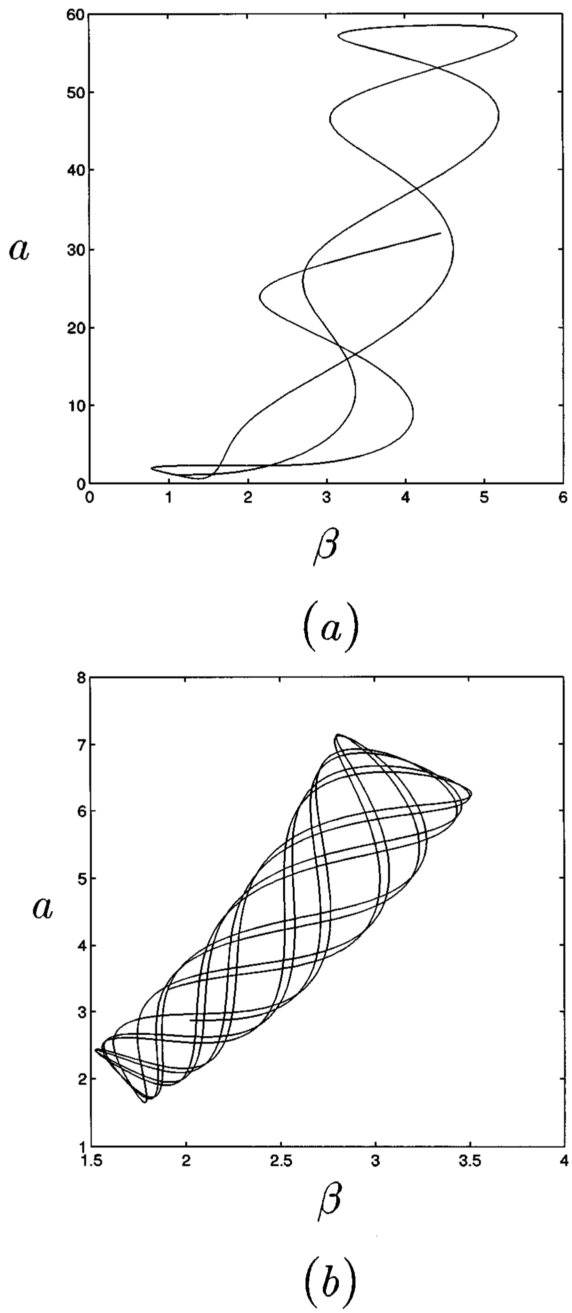

FIG. 4. Set of dynamical trajectories of the system (14) and (15) for $\lambda=0.3$ and $\lambda=-0.3$ and different v_0 with $W=C=1$; (a) $\lambda=-0.3, v_0=0.8$; (b) $\lambda=0.3, v_0=0.2$. The initial point is taken as the position corresponding to the fixed point at $\lambda=0$.

$$\frac{1}{2}m_a v_{\rm crit}^2+\frac{1}{2}m_\beta u_{\rm crit}^2=-U(a_0,\beta_0), \tag{24}$$

where $v_{\rm crit}$ and $u_{\rm crit}$ are the critical velocities in the a and β directions for the dynamical trajectory starting at the FP point (a_0,β_0). In particular, taking $u_{\rm crit}=0$ and considering, for simplicity, the balanced case with $W=C$ we find

$$v_{\rm crit}=\frac{1-\lambda}{\sqrt{2}}W=a_0^{-1}. \tag{25}$$

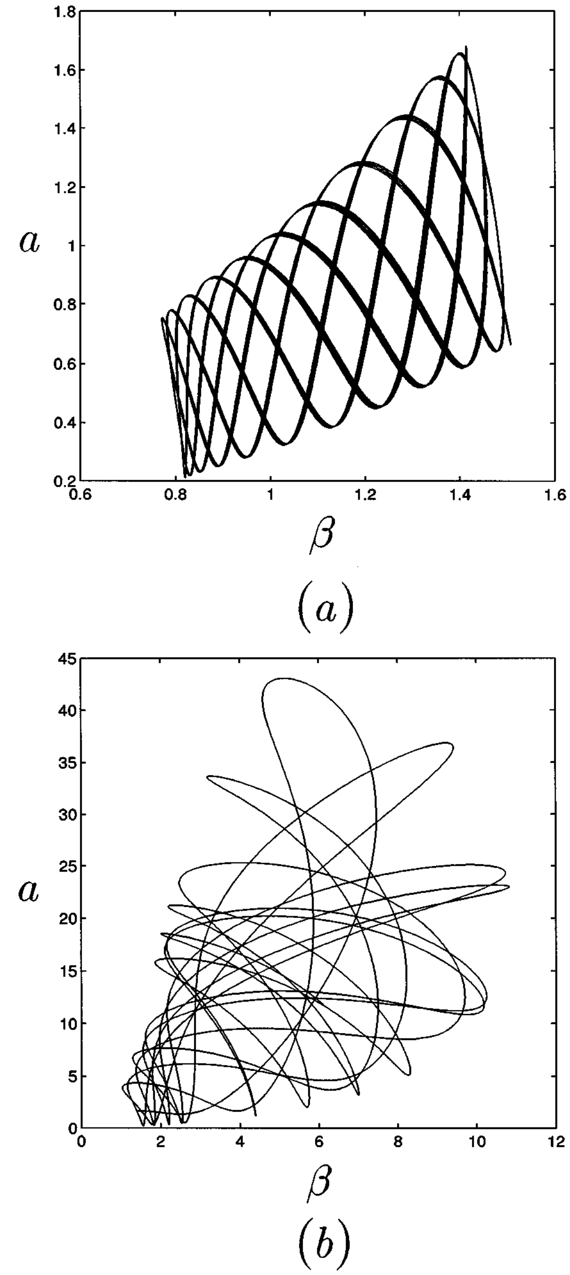

FIG. 5. Dynamical trajectories corresponding to a perturbation with an imbalance in the integrals of motion: $W=2, C=1$, at $\lambda=0$, and the energy (a) $H=-0.5$ and (b) $H=-0.1$.

It is also interesting to consider the oscillations for imbalanced integrals of motion, i.e., for $W\neq C$. As it was discussed in Sec. III, exact solitons always have $W=C$. Therefore, one may expect that, if the initial state is imbalanced, the balance will be gradually restored through emission of radiation. Nevertheless, anticipating that radiation losses are weak, one may expect that the imbalanced perturbation will stay trapped in the soliton for a long time, which makes it meaningful to simulate Eqs. (16) and (17) at $W\neq C$. A typical set of trajectories is shown (for $W=2, C=1$, and $\lambda=0$) in Fig. 5. In this case, one can easily see that an increase of the initial velocity transforms the regular oscillations [Figs. 5(a)] into seemingly irregular ones [Fig. 5(b)], before the particle is kicked out of the potential well.

On the basis of these and many other runs of the simulations, we conclude that finite-amplitude vibrations of the perturbed solitons remain regular if its two integrals of motion, the number of hf quanta and the lf mass, are balanced, but that a sufficiently strong perturbation that introduces an imbalance between them renders the vibrations irregular and possibily chaotic.

V. CONCLUSION

In this work, we have presented a detailed analysis of the internal vibrations of the soliton in a fundamental nonintegrable model, the generalized Zakharov system, which is a universal model of interaction between high- and low-frequency waves in one dimension. The analysis is based on the variational approach. Adaptation of the variational approach to this model is in itself a nontrivial technical problem, as it is necessary to circumvent a divergence that one encounters when applying the variational technique in the traditional way. Eventually, a Hamiltonian system with two degrees of freedom has been derived to govern the vibrations of the soliton. The eigenfrequencies of the small oscillations around the fixed point (which corresponds to the unperturbed soliton) have been found. They are always real, i.e., there is no instability of the soliton. One eigenfrequency lies inside the gap of the high-frequency subsystem, while another is well above the gap.

Large-amplitude oscillations have been simulated numerically. It was demonstrated that, if the perturbed soliton keeps balance between the values of the two fundamental integrals of motion (the number of high-frequency quanta) and low-frequency mass) its vibrations remain regular (periodic or quasiperiodic). However, a perturbation introducing an imbalance between the two conserved quantities renders the vibrations irregular and possibly chaotic.

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